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LETTER TO THE EDITOR

Comment on "On mixed Variational Formulations of Linear Elasticity Using Nonsymmetric Stresses and Displacement", *Int. J. Solids Structures* Vol. 34, No. 10, pp. 1283–1292 (1997), by E. Bertóti

The recent article of Bertóti (1997) deals with the two-field variational principles for the formulation of finite element methods in solid mechanics. Presented formulations of the elasticity problem define the solution as a saddle point of certain functionals involving both displacements and stresses. The paper gives a derivation of Hellinger–Reissner principle in terms of nonsymmetric stresses and displacements, and a new interpretation of Fraeijs de Veubeke's principle. The usual way to enforce the rotational equilibrium equations into the dual mixed variational principle is achieved through introducing the rotations as Lagrange multipliers. This approach has been used by Reissner (1965) and Fraeijs de Veubeke (1972, 1975). The paper of Bertóti (1997) presents a different way as the application of Lagrange multipliers for infliction of symmetry requirement on the stress tensor assumed to be not a priori symmetric. The main idea in his approach is to look for the symmetry-equivalent conditions for the stress tensor, or more precisely conditions that assure the vanishing of the skew-symmetric part of the nonsymmetric stress tensor. The symmetry-equivalent conditions derived in Section 3 incorporated into dual mixed variational principles using the displacement only as Lagrange multipliers, the introduction of the rotations as additional Lagrange multipliers is not needed. This new approach leads to a two-field variational formulation instead of a three-field one.

My comment is dealing with the symmetry-equivalent conditions for the stress tensor which is formulated in Lemma 1 of Section 3. I wish to point out in this comment that Section 3 of Bertóti's paper (Bertóti, 1997) contains a noncorrect statement. Throughout this comment we will use the notations of Bertóti except the example where some new quantities and variables will be introduced.

Consider the resolution of the nonsymmetric stress tensor $\mathbf{T}$ into its symmetric and skew-symmetric parts:

$$\mathbf{T} = \boldsymbol{\sigma} + \boldsymbol{\tau}, \tag{1}$$

$$\boldsymbol{\sigma} = \operatorname{symm} \mathbf{T} = \tfrac{1}{2}(\mathbf{T} + \mathbf{T}^{\mathrm{T}}), \tag{2}$$

$$\boldsymbol{\tau} = \operatorname{skew} \mathbf{T} = \tfrac{1}{2}(\mathbf{T} - \mathbf{T}^{\mathrm{T}}). \tag{3}$$

Let the region Ω be occupied by an elastic body in the three-dimensional space. Let Ω be bounded by sufficiently smooth boundary Γ and let the outward unit normal vector of boundary surface Ω be devoted by $\mathbf{n}$. The Lemma 1 states that:

Let τ be divergence-free in Ω and let its traction be zero vector on the whole boundary surface Γ, i.e.

$$\tau^{kl}_{;k} = 0 \quad \text{in } \Omega, \tag{4}$$

$$n_k \tau^{kl} = 0 \quad \text{on } \Gamma. \tag{5}$$

Then

$$\tau^{kl} = 0 \quad \text{in } \Omega. \tag{6}$$

In the next, I will prove that this statement is incorrect. Equations (4), (5) in a Cartesian coordinate system x, y, z can be read as

$$\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z} = 0 \quad \text{in } \Omega, \tag{7a}$$

$$-\frac{\partial c}{\partial x} + \frac{\partial a}{\partial z} = 0 \quad \text{in } \Omega, \tag{7b}$$

$$\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} = 0 \quad \text{in } \Omega; \tag{7c}$$

$$cn_y - bn_z = 0 \quad \text{on } \Gamma \tag{8a}$$

$$-cn_x + an_z = 0 \quad \text{on } \Gamma \tag{8b}$$

$$bn_x - an_y = 0 \quad \text{on } \Gamma. \tag{8c}$$

Here, we have used the following designations:

$$\mathbf{n} = n_x \mathbf{e}_x + n_y \mathbf{e}_y + n_z \mathbf{e}_z, \tag{9}$$

$\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ are the unit vectors along the axes x, y, z respectively,

$$\tau_{32} = -\tau_{23} = a, \tag{10a}$$

$$\tau_{13} = -\tau_{31} = b, \tag{10b}$$

$$\tau_{21} = -\tau_{12} = c. \tag{10c}$$

According to the skew-symmetry of tensor $\boldsymbol{\tau}$ the main diagonal elements of $\boldsymbol{\tau}$ are vanishing that is

$$\tau_{11} = \tau_{22} = \tau_{33} = 0. \tag{11}$$

The following example of this comment will demonstrate the statement of Lemma 1 is false.

Counter-example

Let Ω be a spherical domain of radius R which is given by the prescription

$$(x, y, z) \in \Omega \quad \text{if } x^2 + y^2 + z^2 - R^2 < 0. \tag{12}$$

It is evident that the equation of bounding surface Γ in the present example is

$$x^2 + y^2 + z^2 - R^2 = 0, \tag{13}$$

and

$$n_x = \frac{x}{R}, \quad n_y = \frac{y}{R}, \quad n_z = \frac{z}{R}. \tag{14a,b,c}$$

Consider the next functions of three variables x, y, z as a triplet a, b, c to determine a skew-symmetric tensor $\boldsymbol{\tau}$:

$$a = a(x,y,z) = K(x^2+y^2+z^2-R^2)x, \tag{15a}$$

$$b = b(x,y,z) = K(x^2+y^2+z^2-R^2)y, \tag{15b}$$

$$c = c(x,y,z) = K(x^2+y^2+z^2-R^2)z. \tag{15c}$$

Here, K is a constant different from zero. A simple calculation yields the result:

$$\frac{\partial a}{\partial y} = 2Kxy, \quad \frac{\partial a}{\partial z} = 2Kxz, \tag{16a,b}$$

$$\frac{\partial b}{\partial x} = 2Kxy, \quad \frac{\partial b}{\partial z} = 2Kyz, \tag{17a,b}$$

$$\frac{\partial c}{\partial x} = 2Kxz, \quad \frac{\partial c}{\partial y} = 2Kyz, \tag{18a,b}$$

$$a = b = c = 0 \quad \text{on } \Gamma. \tag{19}$$

Using Eqns (16), (17), (18), (19) we can check that the skew-symmetric tensor field

$$\tau_{12} = -\tau_{21} = -c = -K(x^2+y^2+z^2-R^2)z, \tag{20a}$$

$$\tau_{13} = -\tau_{31} = b = K(x^2+y^2+z^2-R^2)y, \tag{20b}$$

$$\tau_{23} = -\tau_{32} = -a = -K(x^2+y^2+z^2-R^2)x, \tag{20c}$$

$$\tau_{11} = \tau_{22} = \tau_{33} = 0 \tag{20d}$$

satisfies all conditions of Lemma 1. We have constructed such skew-symmetrical tensor field which is divergence-free in Ω and traction-free on Γ but it is not identical to zero tensor field in Ω. Existence of this tensor field is inconsistent with the statement of Lemma 1 thus Lemma 1 formulates a false statement.

The generalization of Hellinger–Reissner principle and modification of Fraeijs de Veubeke's principle using the nonsymmetric stress and displacement fields are based on the "validity" of Lemma 1 (Bertóti, 1997). May it be that their derivation contains some errors?

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